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Separability dynamics of two-mode Gaussian states in parametric conversion and amplification

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Abstract

We give a simplified form of Simon's separability criterion for two-mode Gaussian states, showing that for systems whose unitary evolution is governed by arbitrary time-dependent quadratic Hamiltonians, the separability dynamics is completely described in terms of the determinant of the cross-covariance matrix. As concrete examples, we consider the evolution of the 'inverse negativity coefficient' (which gives a quantitative estimation of the 'degree of entanglement') for two initially uncoupled modes (each being in a squeezed thermal state) in the cases of parametric converter, parametric amplifier and for a cavity whose boundary oscillates in resonance with two field modes.

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1. Introduction

Various problems related to *entangled quantum states* were subjects of numerous studies performed over the past decade [1–4]. One of them is the condition of *separability* of mixed quantum states, i.e., a possibility of representing the statistical operator $\hat{\rho}$ of the total system as a sum of direct products of statistical operators acting on each part separately:

$$\hat{\rho} = \sum_i p_i \hat{\rho}_{i1} \otimes \hat{\rho}_{i2} \qquad p_i \geqslant 0 \quad \sum_i p_i = 1. \tag{1}$$

Recently, this problem was solved for bipartite continuous variable *Gaussian* states [4–11]. In the most explicit form, the separability criterion was given by Simon [6].

The aim of our paper is to show that Simon's criterion can be significantly simplified, if the system under consideration does not interact with any dissipative environment, and its dynamics is governed by an arbitrary *quadratic* Hamiltonian. It turns out that this criterion is closely related to the concept of *universal quantum invariants* introduced in [12, 13]. For this

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reason, instead of calculating determinants and traces of several matrices and their products in accordance with the initial formulation [6], it is sufficient to calculate the determinant of the only matrix composed from cross-covariances between quadrature components of two modes. This is demonstrated in section 2, where we introduce, beside the 'separability parameter', an 'inverse negativity coefficient', which can be used, as well as other 'negativities' [14–16], for quantitative estimations of the 'degree of entanglement'.

In sections 3–5 we consider, as examples, the evolution of the separability parameter and inverse negativity for different mechanisms of entanglement: parametric conversion (section 3) and parametric amplification. In the latter case, we compare two types of parametric excitation: an external time-dependent pumping in a cavity with fixed geometry (section 4) and the resonance between oscillating boundary and field modes in a cavity with specific spectrum of the unperturbed field eigenfrequencies (section 5). Section 6 contains a discussion of the results obtained.

2. Simplified separability criterion and measures of (in)separability

We consider two-mode continuous variable systems, which can be described in terms of standard bosonic annihilation/creation operators $\hat{a}_k, \hat{a}_k^\dagger$, or equivalent quadrature components operators (we assume $\hbar = 1$):

$$\hat{a}_k = (\omega_k \hat{x}_k + \mathrm{i}\hat{p}_k)/\sqrt{2\omega_k} \qquad k = 1, 2. \tag{2}$$

It is well known that *Gaussian states* are completely determined by mean values and (co)variances of the operators (2) [4, 17–20]. We assume for simplicity that all mean values are equal to zero (otherwise it is sufficient to replace the operators $\hat{a}_k$ by $\hat{a}_k - \langle \hat{a}_k \rangle$). Then symmetrical real covariances are defined as $q_{\alpha\beta} \equiv \frac{1}{2}\langle \hat{q}_\alpha \hat{q}_\beta + \hat{q}_\beta \hat{q}_\alpha \rangle$, where q_α are components of the four-dimensional vector $\mathbf{q} = (x_1, p_1, x_2, p_2)$. It is convenient to gather the covariances in the symmetrical 4×4 covariance matrix $\mathcal{Q}$, splitting this matrix in 2×2 blocks as follows:

$$\mathcal{Q} = \|q_{\alpha\beta}\| = \begin{Vmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{Vmatrix}. \tag{3}$$

These blocks possess the properties $\mathcal{Q}_{11} = \tilde{\mathcal{Q}}_{11}$, $\mathcal{Q}_{22} = \tilde{\mathcal{Q}}_{22}$, $\mathcal{Q}_{12} = \tilde{\mathcal{Q}}_{21}$, where the tilde over a matrix means matrix transposition.

Gaussian operator exponentials cannot be represented as *finite* sums of the form (1). However, if a replacement of the sum by an integral is permitted, i.e., if *continuous decompositions* over *infinite number* of operator products are admissible, then certain families of Gaussian states become separable. It was shown [5, 6, 8, 10] that *the continuous separability of a Gaussian state is equivalent to its 'classicality'*, in the sense of possessing well-defined Sudarshan–Glauber *P*-distribution. For our purposes the most convenient criterion of separability is that found in [6], because it is expressed directly in terms of invariants of blocks of the covariance matrix (3). Namely, the necessary and sufficient condition of separability of a Gaussian state possessing the covariance matrix $\mathcal{Q}$ is the inequality

$$I_1 I_2 + (|I_3| - 1/4)^2 - I_4 \geqslant (I_1 + I_2)/4 \tag{4}$$

where

$$I_1 = \det \mathcal{Q}_{11} \qquad I_2 = \det \mathcal{Q}_{22} \qquad I_3 = \det \mathcal{Q}_{12}$$
$$I_4 = \mathrm{Tr}(\mathcal{Q}_{11}\Sigma\mathcal{Q}_{12}\Sigma\mathcal{Q}_{22}\Sigma\mathcal{Q}_{21}\Sigma) \qquad \Sigma = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

For non-Gaussian states inequality (4) is only a *necessary* condition for separability [6].

Taking into account that the term I_4, given by the trace of the product of eight matrices, is in fact incorporated in the determinant of the total covariance matrix, due to the identity [6]

$$\det \mathcal{Q} = I_1 I_2 + I_3^2 - I_4,$$

the separability criterion (4) can be written in a simpler form (see also [8]) $\mathcal{S} \geqslant 0$, where

$$\mathcal{S} = \det \mathcal{Q} + \tfrac{1}{16} - \tfrac{1}{4}(\det \mathcal{Q}_{11} + \det \mathcal{Q}_{22}) - \tfrac{1}{2}|\det \mathcal{Q}_{12}|. \tag{5}$$

Now let us suppose that the system's dynamics is governed by some Hamiltonian which is a *quadratic form* of operators (2) with arbitrary (in general, time-dependent) coefficients. It was discovered in [12] (see also [13, 19] for generalizations) that such systems possess *universal quantum invariants*, i.e., certain combinations of variances which are conserved in time *independently of a concrete form of coefficients of the Hamiltonian*. These invariants exist due to the symplectic structure of the transformation relating initial and time-dependent values of the quadrature components operators. In the two-mode case there are two independent universal invariants, directly connected with the terms of equation (5) [12, 13]:

$$\mathcal{D}_0 = \det \mathcal{Q} \qquad \mathcal{D}_2 = \det \mathcal{Q}_{11} + \det \mathcal{Q}_{22} + 2\det \mathcal{Q}_{12}. \tag{6}$$

Consequently,

$$\mathcal{S}(t) = \mathcal{S}(0) + \tfrac{1}{2}(\det \mathcal{Q}_{12}(t) - |\det \mathcal{Q}_{12}(t)|) \tag{7}$$

where

$$\mathcal{S}(0) = \mathcal{D}_0 - \tfrac{1}{4}\mathcal{D}_2 + \tfrac{1}{16} \tag{8}$$

is nonnegative due to the generalized uncertainty relations [12, 19]. Thus we see, indeed, that the separability of Gaussian states of bipartite quantum systems, whose evolution is *unitary* (even if the Hamiltonian is time dependent), is determined completely by the determinant of the cross-covariance matrix $\det \mathcal{Q}_{12}(t)$ (for given initial conditions). In particular, it becomes quite obvious from the form (7) that the *necessary* (although not sufficient) condition of inseparability ('entanglement') is [6] $\det \mathcal{Q}_{12} < 0$.

The 'separability parameter' $\mathcal{S}(t)$ can assume, in principle, any value in the infinite interval $(-\infty, \infty)$. One could wish to have some *compact* parameter characterizing the degree of (in)separability in such a way, that its values would be confined within the interval $(-1, 1)$, so that negative values would correspond to inseparable states (in some analogy with Mandel's parameter of 'nonclassicality'), while positive values would correspond to separable ('classically correlated') states. Of course, the choice of such a parameter is not unique: any monotonous function $f(\mathcal{S})$ satisfying the condition $-1 < f(\mathcal{S}) < 1$ could serve for this purpose. Simple examples are, e.g., the functions

$$\mathcal{A}(\mathcal{S}) = \tanh(\alpha \mathcal{S}) \tag{9}$$

$$\mathcal{B}(\mathcal{S}) = \operatorname{sign}(\mathcal{S})\big(1 + |\beta \mathcal{S}| - \sqrt{1 + (\beta \mathcal{S})^2}\big) \tag{10}$$

where α and β are some positive constant coefficients.

However, in order to follow current trends in studies on entanglement, we prefer to use a function which is close to the so-called 'negativity' [14–16]. In the case of Gaussian states the 'logarithmic negativity' is defined by the formula

$$\mathcal{E}_N = \sum_{k=1}^{2} F(|c_k|) \qquad F(x) = \begin{cases} 0, & 2x \geqslant 1 \\ -\log_2(2x), & 2x < 1 \end{cases} \tag{11}$$

where the arguments c_k are the so-called 'symplectic eigenvalues' of the 'partially transposed' variance matrix $\mathcal{Q}^{(\mathrm{PT})}$, which is obtained from matrix (3) by changing the sign of the 'momentum-coordinate' covariances in matrices $\mathcal{Q}_{12}$ and $\mathcal{Q}_{21}$ (this procedure corresponds to the reflection of momentum of one subsystem of the bipartite system, e.g., a transformation of the vector $\mathbf{q}$ of the form $\mathbf{q} \to (x_1, -p_1, x_2, p_2)$). The symplectic eigenvalues of the symmetrical matrix $\mathcal{Q}$ are defined as eigenvalues of the matrix $\mathcal{X} = \mathcal{Q}\mathcal{Z}^{-1}$, where the 4×4 antisymmetrical matrix $\mathcal{Z}$ consists of commutators between the elements of vector $\mathbf{q}$. In our case,

$$\mathcal{Z} = \mathrm{i}\begin{Vmatrix} \Sigma & 0 \\ 0 & \Sigma \end{Vmatrix}.$$

The sum in (11) contains only two terms because the set of symplectic eigenvalues consists of pairs $\pm c_k$, $k = 1, 2$. One of the reasons for definition (11) is that for separable states the change $\mathcal{Q} \to \mathcal{Q}^{(\mathrm{PT})}$ results in a new variance matrix corresponding to another physical state, and the inequality $2|c_k| \geqslant 1$ is one of many apparently different forms of uncertainty relations, while for inseparable states partial transpositions result in covariance matrices which cannot be related to any physical state (thus violating the uncertainty relations). An explicit form of the symplectic eigenvalues $\kappa_{1,2}$ of *true* 4×4 covariance matrices was obtained in [21] (see also [22, 23]):

$$|\kappa_{1,2}| = \tfrac{1}{2}\left[\sqrt{\mathcal{D}_2 + 2\sqrt{\mathcal{D}_0}} \pm \sqrt{\mathcal{D}_2 - 2\sqrt{\mathcal{D}_0}}\right] \tag{12}$$

where invariants $\mathcal{D}_0$ and $\mathcal{D}_2$ are defined in (6). The characteristic equation for matrix $\mathcal{Q}^{(\mathrm{PT})}\mathcal{Z}^{-1}$ has the same form as for matrix $\mathcal{Q}\mathcal{Z}^{-1}$, with the only difference that one should change the sign of $\det \mathcal{Q}_{12}$. This means that the values $|c_{1,2}|$ can be obtained from $|\kappa_{1,2}|$ (12) by means of substitution [14–16]

$$\mathcal{D}_2 \to \tilde{\mathcal{D}}_2 \equiv \det \mathcal{Q}_{11} + \det \mathcal{Q}_{22} - 2\det \mathcal{Q}_{12}. \tag{13}$$

Obviously,

$$\tilde{\mathcal{D}}_2 = \mathcal{D}_2 - 4\det \mathcal{Q}_{12}, \tag{14}$$

and since $\mathcal{D}_0$ and $\mathcal{D}_2$ do not depend on time in the case involved, the dynamics of the logarithmic negativity is also determined completely by the time dependence of the only quantity $\det \mathcal{Q}_{12}$. In order to deal with compact measures, we shall study instead of $\mathcal{E}_N$ the 'inverse negativity'

$$\mathcal{I} = 2^{-\mathcal{E}_N} - 1 = -2\mathcal{N}/(2\mathcal{N} + 1) \tag{15}$$

where $\mathcal{N} = \frac{1}{2}(2^{\mathcal{E}_N} - 1)$ is the 'negativity' introduced in [14]. The values of the coefficient $\mathcal{I}$ are close to -1 for strongly entangled states, while $\mathcal{I} \equiv 0$ for separable states. Now let us note that the invariant $\mathcal{D}_0$ is connected with symplectic eigenvalues as $\mathcal{D}_0 = |\kappa_1\kappa_2|^2 = |c_1c_2|^2$ (because partial transposition does not change $\det \mathcal{Q}$). Then an immediate consequence of the generalized uncertainty relation $\det \mathcal{Q} \geqslant 1/16$ [12, 17, 19, 24] is that at least one of the symplectic eigenvalues must exceed the value $1/2$. This means that the sum in (11) contains in fact only one term with $|c_{\min}|$. Therefore, the inverse negativity can be written as

$$\mathcal{I} = \min(0, 2|c_{\min}| - 1) \tag{16}$$

with

$$2|c_{\min}| = \sqrt{\tilde{\mathcal{D}}_2 + 2\sqrt{\mathcal{D}_0}} - \sqrt{\tilde{\mathcal{D}}_2 - 2\sqrt{\mathcal{D}_0}}. \tag{17}$$

The following expressions hold for the determinants of the covariance and cross-covariance block matrices (remember that we suppose that all average values of the quadrature components are zero):

$$\det \mathcal{Q}_{kk} = \langle \hat{x}_k^2 \rangle \langle \hat{p}_k^2 \rangle - \tfrac{1}{4} \langle \hat{x}_k \hat{p}_k + \hat{p}_k \hat{x}_k \rangle^2 = \tfrac{1}{4} \langle \hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \rangle^2 - \left| \langle \hat{a}_k^2 \rangle \right|^2 \tag{18}$$

$$\det \mathcal{Q}_{12} = \langle \hat{x}_1 \hat{x}_2 \rangle \langle \hat{p}_1 \hat{p}_2 \rangle - \langle \hat{x}_1 \hat{p}_2 \rangle \langle \hat{p}_1 \hat{x}_2 \rangle = \left| \langle \hat{a}_1 \hat{a}_2^\dagger \rangle \right|^2 - |\langle \hat{a}_1 \hat{a}_2 \rangle|^2. \tag{19}$$

It is worth noting also that the determinants of matrix $\mathcal{Q}$ and its diagonal blocks $\mathcal{Q}_{kk}$ are related to the *purities* of the whole system and its subsystems according to the relations [19]

$$\mu \equiv \mathrm{Tr}\, \hat{\rho}^2 = (16 \det \mathcal{Q})^{-1/2} \qquad \mu_k \equiv \mathrm{Tr}\, \hat{\rho}_k^2 = (4 \det \mathcal{Q}_{kk})^{-1/2}. \tag{20}$$

Any one-mode Gaussian state with zero mean values of quadrature components can be considered as being obtained from some thermal state, defined with respect to 'bare' bosonic operators $\hat{b}, \hat{b}^\dagger$, possessing the second-order average values $\langle \hat{b}^\dagger \hat{b} \rangle = \nu \geqslant 0$ and $\langle \hat{b}^2 \rangle = 0$, by means of a 'dressing' linear canonical transformation of the form [19, 25]

$$\hat{a} = \hat{b} \cosh(r) + \hat{b}^\dagger \sinh(r)\, \mathrm{e}^{\mathrm{i}\chi} \tag{21}$$

where positive coefficient r characterizes the degree of *squeezing* of the 'initial' thermal state and phase χ is responsible for the statistical correlations between the quadrature components in the 'dressed' state (we have suppressed a possible but unsignificant overall phase). We assume for simplicity hereafter that the joint density matrix of two modes is totally disentangled (factorized) at the initial instant $t = 0$, i.e., $\mathcal{Q}_{12}(0) = 0$. Therefore, we shall use the following parametrization of the second-order moments in the initial Gaussian state of the kth mode (assuming that $\langle \hat{a}_k \rangle = 0$ and introducing parameters $\vartheta_k \equiv \nu_k + 1/2 \geqslant 1/2$):

$$\langle \hat{a}_k^\dagger \hat{a}_k \rangle = \vartheta_k \cosh(2r_k) - \tfrac{1}{2} \qquad \langle \hat{a}_k^2 \rangle = \vartheta_k \sinh(2r_k)\, \mathrm{e}^{\mathrm{i}\chi_k}. \tag{22}$$

Due to equation (18), parameters ϑ_k determine purities of each subsystem $\mu_k = (2\vartheta_k)^{-1}$, which do not depend on the degree of squeezing. According to equations (8) and (18), the initial value $\mathcal{S}(0)$ also does not depend on the squeezing parameters for factorized states with $\mu = \mu_1 \mu_2$:

$$\mathcal{S}(0) = \left(\vartheta_1^2 - 1/4\right)\left(\vartheta_2^2 - 1/4\right). \tag{23}$$

However, the parameters r_k and χ_k do influence the evolution of the 'separability coefficients' and the inverse negativity $\mathcal{I}$ through the time dependence of $\det \mathcal{Q}_{12}$. In particular, using equation (14) we can write

$$2|c_{\min}| = \sqrt{(\vartheta_1 + \vartheta_2)^2 - 4 \det \mathcal{Q}_{12}} - \sqrt{(\vartheta_1 - \vartheta_2)^2 - 4 \det \mathcal{Q}_{12}}. \tag{24}$$

The only solution of equation $2|c_{\min}| = 1$ with fixed values of ϑ_1 and ϑ_2 is $\det \mathcal{Q}_{12} = -\left(\vartheta_1^2 - 1/4\right)\left(\vartheta_2^2 - 1/4\right)$. Since the right-hand side of (24) is monotonous function of $\det \mathcal{Q}_{12}$, the inseparability criteria $\mathcal{S}(t) < 0$ and $\mathcal{I}(t) < 0$ are indeed equivalent. If $\det \mathcal{Q}_{12} < 0$ and $4|\det \mathcal{Q}_{12}| \gg (\vartheta_1 + \vartheta_2)^2$, then the inverse negativity tends to -1 as $\mathcal{I} \approx -1 + \vartheta_1 \vartheta_2 / \sqrt{|\det \mathcal{Q}_{12}|}$.

In the following sections, we study the dynamics of separability parameter $\mathcal{S}(t)$ (7) for parametric converters and amplifiers in the special cases of exact resonance, when the solutions have the simplest explicit forms. The evolution of different measures of entanglement and intermode correlations for two harmonic oscillators with constant frequencies and the most general time-dependent resonance bilinear couplings was considered in [23, 26], and entanglement in a chain of oscillators with time-independent parameters was investigated in [27].

3. Parametric converter

The parametric converter Hamiltonian is

$$\hat{H}_c = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \kappa \hat{a}_1^\dagger \hat{a}_2 \, \mathrm{e}^{\mathrm{i}\eta t} + \kappa^* \hat{a}_2^\dagger \hat{a}_1 \, \mathrm{e}^{-\mathrm{i}\eta t} \tag{25}$$

where we set $\eta = \omega_2 - \omega_1$ (confining ourselves to the simplest case of *exact* resonance). The well-known exact solutions of the Heisenberg equations of motion read [28, 29]

$$\hat{a}_1(t) = \mathrm{e}^{-\mathrm{i}\omega_1 t} \left[\hat{a}_1(0) \cos\tau - \frac{\mathrm{i}\kappa}{|\kappa|} \hat{a}_2(0) \sin\tau \right]$$

$$\hat{a}_2(t) = \mathrm{e}^{-\mathrm{i}\omega_2 t} \left[\hat{a}_2(0) \cos\tau - \frac{\mathrm{i}|\kappa|}{\kappa} \hat{a}_1(0) \sin\tau \right]$$

where $\tau \equiv |\kappa| t$. Then,

$$\det \mathcal{Q}_{12}(\tau) = \frac{1}{4} \sin^2(2\tau) R_c \tag{26}$$

$$R_c = \left\{ \left\langle \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1 \right\rangle^2 - \left| \frac{|\kappa|}{\kappa} \left\langle \hat{a}_1^2 \right\rangle + \frac{\kappa}{|\kappa|} \left\langle \hat{a}_2^2 \right\rangle \right|^2 \right\}_{t=0} = (\vartheta_1 - \vartheta_2)^2 - 4\vartheta_1 \vartheta_2 Y \tag{27}$$

where

$$Y = \sinh^2(r_1 + r_2) - \sinh(2r_1)\sinh(2r_2)\sin^2\phi \qquad \phi = \arg(\kappa) + \tfrac{1}{2}(\chi_2 - \chi_1). \tag{28}$$

We see that R_c and, consequently, $\det \mathcal{Q}_{12}(\tau)$ can be negative only provided $\langle \hat{a}_k^2 \rangle(0) \neq 0$ (or $r_k > 0$) at least for one of two values $k = 1, 2$. This means that at least one mode must be initially in a 'nonclassical' state in order that inseparability could arise. For initial thermal states of both modes, these modes cannot become truly entangled in the process of evolution. The parameters r_k must exceed some critical values, in order that inseparability could be achieved. For example, if the first mode was initially in a *pure* state ($\nu_1 = 0$) and the second mode was in a thermal state ($r_2 = 0$), then inseparability can be achieved for any nonzero value of the scaled time τ, if $\nu_2^2 - 2\vartheta_2 \sinh^2(r_1) < 0$, or it cannot be achieved at all, if this expression is positive.

In a generic case the coefficient R_c depends essentially on the phase difference ϕ. In the case $\phi = 0$, which is the most favourable for entanglement, the necessary condition of inseparability (which can be achieved at least at the instant $\tau = \pi/4$) is

$$\sinh(r_1 + r_2) > \frac{\vartheta_1 \vartheta_2 - 1/4}{\sqrt{\vartheta_1 \vartheta_2}}. \tag{29}$$

Now, it is worth remembering that the minimal value of the variances of quadrature components of the family of operators $\hat{a}\, \mathrm{e}^{\mathrm{i}\gamma}$ for $0 \leqslant \gamma < 2\pi$ is given by a simple formula

$$\sigma_{\min} = \tfrac{1}{2} + \langle \hat{a}^\dagger \hat{a} \rangle - |\langle \hat{a}^2 \rangle| \tag{30}$$

(it is known under the names 'principal squeezing' [30] or 'invariant squeezing' [31, 32]). In the case of parametrization (22), formula (30) assumes the form

$$\sigma_{\min}^{(k)} = \vartheta_k \exp(-2r_k) \tag{31}$$

and one can verify that the inequality (29) is equivalent to the inequality $\sigma_{\min}^{(1)} \sigma_{\min}^{(2)} < 1/4$. Consequently, inseparability by means of parametric conversion can be achieved only if at least one mode was initially in truly squeezed state, with the minimal value of one of quadrature components less than the variance of the coherent state $1/2$.

However, the transition from separable to inseparable state is not correlated with the behaviour of physical observables, such as the difference of the mean number of quanta in two modes $\Delta\mathcal{N} = \langle \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 \rangle$, or geometrical characteristics of the state in the Hilbert space, such as the difference of squares of inverse purities of the modes $\Delta\mathcal{M} = \mu_1^{-2} - \mu_2^{-2}$. In the simplest special case of $\chi_1 = \chi_2 = \phi = 0$, both functions have the same time dependences: $\Delta\mathcal{N}(\tau) = \Delta\mathcal{N}(0)\cos(2\tau)$ and $\Delta\mathcal{M}(\tau) = \Delta\mathcal{M}(0)\cos(2\tau)$, whereas $\mathcal{S}(\tau) = \mathcal{S}(0) + R_c \sin^2(2\tau)/4$. At the moments of transition τ_*, determined by the equation $\sin^2(2\tau_*) = -4\mathcal{S}(0)/R_c$, the functions $\Delta\mathcal{N}(\tau)$ and $\Delta\mathcal{M}(\tau)$ do not show any changes in their behaviours. In particular, they can be positive, when the separable state becomes inseparable, but they can be negative, at the moment of the next transformation, from inseparable to separable state.

4. Parametric amplifier

The parametric amplifier Hamiltonian is (again in the case of exact resonance)

$$\hat{H}_a = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \kappa \hat{a}_1 \hat{a}_2 \, \mathrm{e}^{\mathrm{i}\eta t} + \kappa^* \hat{a}_2^\dagger \hat{a}_1^\dagger \, \mathrm{e}^{-\mathrm{i}\eta t} \tag{32}$$

with $\eta = \omega_2 + \omega_1$. The solutions of the Heisenberg equations are [28, 33]

$$\hat{a}_1(t) = \mathrm{e}^{-\mathrm{i}\omega_1 t}\left[\hat{a}_1(0)\cosh\tau - \frac{\mathrm{i}|\kappa|}{\kappa}\hat{a}_2^\dagger(0)\sinh\tau\right]$$

$$\hat{a}_2(t) = \mathrm{e}^{-\mathrm{i}\omega_2 t}\left[\hat{a}_2(0)\cosh\tau - \frac{\mathrm{i}|\kappa|}{\kappa}\hat{a}_1^\dagger(0)\sinh\tau\right]$$

with formally the same scaled time $\tau \equiv |\kappa| t$. Now,

$$\det \mathcal{Q}_{12}(\tau) = \frac{1}{4}\sinh^2(2\tau) R_a \tag{33}$$

$$R_a = \left\{ \left| \frac{|\kappa|}{\kappa}\langle \hat{a}_1^2 \rangle - \frac{\kappa}{|\kappa|}\langle \hat{a}_2^2 \rangle \right|^2 - \langle \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_1^\dagger \rangle^2 \right\}_{t=0} = -(\vartheta_1 + \vartheta_2)^2 - 4\vartheta_1\vartheta_2 Y \tag{34}$$

with Y given in (28). We see that R_a is *always negative*, even for initial thermal states with $r_1 = r_2 = 0$. Consequently, *any initially uncoupled states become inseparable in the process of parametric amplification after sufficiently long time* (due to the factor $\sinh^2(2\tau)$ in equation (33)). According to equation (24), the inverse negativity coefficient goes monotonously to -1 as $\tau \to \infty$.

Let us consider a simple example of initially unsqueezed thermal states with equal temperatures: $\vartheta_1 = \vartheta_2 \equiv \vartheta$, $r_1 = r_2 = 0$. Then $\mathcal{S}(\tau) = (\vartheta^2 - 1/4)^2 - \vartheta^2\sinh^2(2\tau)$, so that the mixed state of two initially uncorrelated modes becomes inseparable at the moment τ_*, when $\sinh(2\tau_*) = (\vartheta^2 - 1/4)/\vartheta$. On the other hand, the mean energy of each mode grows with time as $\mathcal{E}_k(\tau) = \omega_k\vartheta\cosh(2\tau)$, and nothing remarkable happens with this function at the moment $\tau = \tau_*$, when $\mathcal{E}_k(\tau_*) = \omega_k(\vartheta^2 + 1/4)$. Note that the state of each mode remains unsqueezed for any moment of time, as soon as $\langle \hat{a}_1^2 \rangle(t) = \langle \hat{a}_2^2 \rangle(t) = 0$ for the chosen initial conditions.

5. Two resonantly coupled modes in a vibrating cavity

It is worth noting that parametric amplification is not always accompanied with transforming initial factorized states into inseparable entangled states, even in the cases when the energies of each mode grow unlimitedly. An interesting counterexample is the special case of the field in a cavity with moving boundaries, where entanglement between discrete modes occurs due to the Doppler effect. Various aspects of this problem (modification of Casimir's force, photon creation from vacuum, etc) were studied in numerous publications reviewed in [34]. In particular, the problem of entanglement was considered in [35–38].

Strictly speaking, all modes are coupled in this case. However, there exists an important special case of resonance coupling, where only *two* field modes are coupled, whereas interaction with other modes can be neglected (or taken into account as small perturbations) [39].

Confining ourselves to the simplest case of exact resonance, we suppose that one of the cavity's walls performs small oscillations with the frequency $2\omega_1^{(0)}$, where $\omega_1^{(0)}$ is the frequency of some electromagnetic field mode of the stationary cavity. Assuming that $\omega_1^{(0)} \equiv 1$ (i.e., normalizing all frequencies by $\omega_1^{(0)}$), we can write the *instantaneous time-dependent frequency* $\omega_1(t)$ of the excited mode in a cavity with oscillating wall as

$$\omega_1(t) = 1 + 2\epsilon\cos(2t) \qquad |\epsilon| \ll 1. \tag{35}$$

Besides, we suppose that the unperturbed field frequency spectrum includes the frequency $\omega_3^{(0)} = 3$, but it does not contain frequencies close to $5\omega_1^{(0)}$. The possibility of such a situation was pointed out in [39]. An example is a cubic cavity with the pair of modes {111} and {511}. Another example is the pair of modes {110} and {510} in the rectangular cavity with $L_x = \sqrt{2}L_y$ (in this case, the common direction of polarization is along the z-axis). Then we have two parametrically excited and resonantly interacting modes, whose dynamics is governed by the effective Hamiltonian [40]

$$H_{13} = \tfrac{1}{2}\left(p_1^2 + p_3^2\right) + 3\tilde{\mu}\epsilon\sin(2t)(p_1x_3 - p_3x_1) + \tfrac{1}{2}[1 + 4\epsilon\cos(2t)]x_1^2 + \tfrac{9}{2}x_3^2 \tag{36}$$

where symbols x_k and p_j with $k, j = 1, 3$ stand for the quadrature components of the first and third field modes, and $\tilde{\mu}$ is a constant factor, which depends on the cavity geometry (for a cubic cavity, $\tilde{\mu} = 5/12$). An equivalent expression in terms of the annihilation/creation operators is as follows (we omit here nonresonant terms)

$$H_{13} = \hat{a}_1^\dagger\hat{a}_1 + 3\hat{a}_3^\dagger\hat{a}_3 + \epsilon\cos(2t)\left(\hat{a}_1^{\dagger 2} + \hat{a}_1^2\right) + \mathrm{i}\sqrt{12}\tilde{\mu}\epsilon\sin(2t)\left(\hat{a}_1^\dagger\hat{a}_3 - \hat{a}_1\hat{a}_3^\dagger\right). \tag{37}$$

The Heisenberg equations of motion for the operators $\hat{x}_k$ and $\hat{p}_k$, following from the Hamiltonian (36), have been solved in the frameworks of the method of slowly varying amplitudes in [40]. In terms of the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ these solutions have the following form:

$$\hat{a}_1(t) = \left\{\hat{a}_1(0)\left[\cos(\zeta\tau)\cosh\tau + \frac{\sin(\zeta\tau)}{\zeta}\sinh\tau\right] - \mathrm{i}\hat{a}_1^\dagger(0)\left[\cos(\zeta\tau)\sinh\tau + \frac{\sin(\zeta\tau)}{\zeta}\cosh\tau\right]\right.$$
$$\left. + \sqrt{2\gamma}\frac{\sin(\zeta\tau)}{\zeta}\left[\hat{a}_3^\dagger(0)\sinh\tau - \mathrm{i}\hat{a}_3(0)\cosh\tau\right]\right\}\mathrm{e}^{-\mathrm{i}t} \tag{38}$$

$$\hat{a}_3(t) = \left\{\hat{a}_3(0)\left[\cos(\zeta\tau)\cosh\tau - \frac{\sin(\zeta\tau)}{\zeta}\sinh\tau\right] + \mathrm{i}\hat{a}_3^\dagger(0)\left[\cos(\zeta\tau)\sinh\tau - \frac{\sin(\zeta\tau)}{\zeta}\cosh\tau\right]\right.$$
$$\left. - \sqrt{2\gamma}\frac{\sin(\zeta\tau)}{\zeta}\left[\hat{a}_1^\dagger(0)\sinh\tau + \mathrm{i}\hat{a}_1(0)\cosh\tau\right]\right\}\mathrm{e}^{-3\mathrm{i}t} \tag{39}$$

where

$$\tau \equiv \tfrac{1}{2}\epsilon t \qquad \zeta = \sqrt{2\gamma - 1} \qquad \gamma \equiv 96\tilde{\mu}^2. \tag{40}$$

For the modes {111} and {511} of the cubic cavity or {110} and {510} of the rectangular cavity with $L_x = \sqrt{2}L_y$ we have $\gamma = 50/3$. Due to this explicit example, we assume that parameter γ is large: $\gamma \gg 1$.

Using equations (19), (22), (38) and (39) (with the change of indices $2 \to 3$) we can express the determinant of the cross-covariance matrix as

$$\det \mathcal{Q}_{13}(\tau) = 2\gamma \frac{\sin^2(\zeta\tau)}{\zeta^2} \Bigg\{ \cos^2(\zeta\tau)[(\vartheta_3 - \vartheta_1)^2 - 4\vartheta_1\vartheta_3 Y_-] \\ - 2\vartheta_1\vartheta_3 Z \frac{\sin(2\zeta\tau)}{\zeta} - \frac{\sin^2(\zeta\tau)}{\zeta^2}[(\vartheta_3 + \vartheta_1)^2 + 4\vartheta_1\vartheta_3 Y_+] \Bigg\} \tag{41}$$

where the coefficients

$$Y_\pm = \sinh^2(r_1 \mp r_3) \pm \sinh(2r_1)\sinh(2r_3)\sin^2\left(\frac{\chi_1 \pm \chi_3}{2}\right)$$

and

$$Z = \sin\chi_1 \sinh(2r_1)\cosh(2r_3) + \sin\chi_3 \sinh(2r_3)\cosh(2r_1)$$

become zero in the special case of initial thermal states of both modes ($r_1 = r_3 = 0$).

We see that the degree of separability is strictly *periodical* function of the 'slow time' τ (note that the interaction part of the Hamiltonian (37) has the same form as for the parametric converter). Nonetheless, mean energies of each mode, $\mathcal{E}_k \equiv \omega_k\left(\langle\hat{a}_k^\dagger\hat{a}_k\rangle + \frac{1}{2}\right)$, increase in time almost exponentially (with some superimposed oscillations):

$$\mathcal{E}_1(\tau) = \vartheta_1 C_1(\tau)\cos^2(\zeta\tau) + \vartheta_1 S_1(\tau)\frac{\sin(2\zeta\tau)}{\zeta} + [\vartheta_1 C_1(\tau) + 2\gamma\vartheta_3 C_3(\tau)]\frac{\sin^2(\zeta\tau)}{\zeta^2} \tag{42}$$

$$\mathcal{E}_3(\tau)/3 = \vartheta_3 C_3(\tau)\cos^2(\zeta\tau) - \vartheta_3 S_3(\tau)\frac{\sin(2\zeta\tau)}{\zeta} + [\vartheta_3 C_3(\tau) + 2\gamma\vartheta_1 C_1(\tau)]\frac{\sin^2(\zeta\tau)}{\zeta^2} \tag{43}$$

where

$$C_1(\tau) = \cosh(2\tau)\cosh(2r_1) - \sinh(2\tau)\sinh(2r_1)\sin\chi_1$$
$$C_3(\tau) = \cosh(2\tau)\cosh(2r_3) + \sinh(2\tau)\sinh(2r_3)\sin\chi_3$$
$$S_1(\tau) = \sinh(2\tau)\cosh(2r_1) - \cosh(2\tau)\sinh(2r_1)\sin\chi_1$$
$$S_3(\tau) = \sinh(2\tau)\cosh(2r_3) + \cosh(2\tau)\sinh(2r_3)\sin\chi_3.$$

For $\zeta \gg 1$, we have simplified approximate expressions

$$\mathcal{E}_1(\tau) \approx \vartheta_1 C_1(\tau)\cos^2(\zeta\tau) + \vartheta_3 C_3(\tau)\sin^2(\zeta\tau)$$
$$\mathcal{E}_3(\tau)/3 \approx \vartheta_3 C_3(\tau)\cos^2(\zeta\tau) + \vartheta_1 C_1(\tau)\sin^2(\zeta\tau).$$

For initial *unsqueezed thermal* states, the minimal values of $\det \mathcal{Q}_{13}$ are achieved when $\zeta\tau = \pi/2 + k\pi, k = 0, 1, \ldots$:

$$\det \mathcal{Q}_{13}^{\min} = -(\vartheta_3 + \vartheta_1)^2(\zeta^2 + 1)/\zeta^4.$$

In the 'high-temperature' case, when $\vartheta_1 \approx 3\vartheta_3 \gg 1$ (due to the energy equipartition law), formula (23) yields $S(0) \approx \vartheta_1^4/9$, whereas $\det \mathcal{Q}_{13}^{\min} \approx -2\vartheta_1^2/\zeta^2$ (for $\zeta \gg 1$). Consequently, in this case a highly mixed quantum state of two *coupled* modes remains separable for all times, although their energies increase unlimitedly (in contradistinction to the case of parametric amplifier considered in section 4).

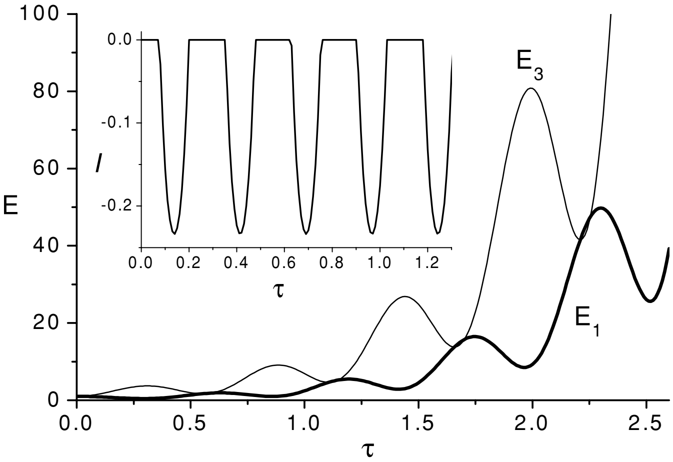

Figure 1. The normalized energies $E_k(\tau) \equiv \mathcal{E}_k(\tau)/\mathcal{E}_k(0)$ of the first and third modes in a cubic cavity with resonantly oscillating boundary ($\gamma = 50/3$) versus the 'slow time' τ, for initial highly squeezed thermal states with $\vartheta_1 = 3\vartheta_3 = 100$, $r_1 = r_3 = 2.5$, and $\chi_1 = \chi_3 = 0$. In the insertion: the inverse negativity coefficient $\mathcal{I}$ (16) between the first and third modes under the same initial conditions.

However, the situation may be quite different for initial *squeezed* states. For example, if $\chi_1 = \chi_3 = 0$, $\vartheta_1 \approx 3\vartheta_3 \gg 1$ and $\zeta \gg 1$, then the minima of function $\det \mathcal{Q}_{13}(\tau)$ happen for $\zeta\tau \approx \pi/4 + m\pi$, when

$$\det \mathcal{Q}_{13}^{\min} \approx \frac{\vartheta_1^2}{9}[1 - 3\sinh^2(r_1 + r_3)].$$

Consequently, inseparability can be achieved, provided the initial degree of squeezing is big enough at least for one state. Taking for simplicity $r_1 = r_3 = r \gg 1$, we obtain the following approximate expressions for mean energies of the two modes:

$$\mathcal{E}_1(\tau) \approx \vartheta_1 \cosh(2\tau)\cosh(2r)\left[\cos^2(\zeta\tau) + \tfrac{1}{3}\sin^2(\zeta\tau)\right] \quad (44)$$

$$\mathcal{E}_3(\tau) \approx \vartheta_1 \cosh(2\tau)\cosh(2r)[\cos^2(\zeta\tau) + 3\sin^2(\zeta\tau)]. \quad (45)$$

From the point of view of energy, interesting things happen at the moments $\tau_n = \pi n/\zeta$, when $\mathcal{E}_1 = \mathcal{E}_3$, and at the moments close to $\tau_n' = (n + 1/2)\pi/\zeta$, when $\mathcal{E}_1$ attains its local minimum and $\mathcal{E}_3$ attains its local maximum with $\mathcal{E}_3^{\max} \approx 9\mathcal{E}_1^{\min}$ (remember that $\zeta \gg 1$): see figure 1. However, at these moments the quantum state is always separable, because the inverse negativity coefficient $\mathcal{I}$ (16) equals exactly zero for $\tau = \tau_n$ and $\tau = \tau_n'$, as one can see in the insertion of figure 1 and from an approximate analytical expression

$$\mathcal{I}(\tau) \approx \min\left\{0, \tfrac{2}{3}\vartheta_1\left[\sqrt{4 + 3\sinh^2(2r)\sin^2(2\zeta\tau)} - \sqrt{1 + 3\sinh^2(2r)\sin^2(2\zeta\tau)}\right] - 1\right\}. \quad (46)$$

The joint state of two modes is inseparable when $|\sin(2\zeta\tau)| > 2/[\sqrt{3}\sinh(2r)]$, and maximal degree of inseparability (entanglement) is achieved at the moments $\tilde{\tau}_k = \pi(k + 1/4)/\zeta$. However, nothing remarkable happens with mean energies in these time intervals.

Time dependences of the mean energies turn out to be rather sensitive to phases of the initial squeezing parameters. For instance, choosing $\chi_1 = \chi_3 = \pi/2$ and the same values $r_1 = r_3 = r$, ζ, ϑ_1 and ϑ_3 as before, we obtain the following approximate expressions instead

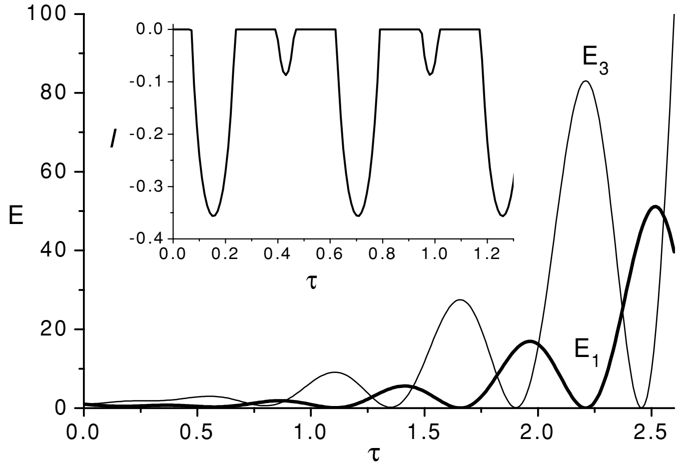

Figure 2. The same as in figure 1, but for $\chi_1 = \chi_3 = \pi/2$.

of (44) and (45):

$$\mathcal{E}_1(\tau) \approx \vartheta_1\left[\cosh(2\tau - 2r)\cos^2(\zeta\tau) + \tfrac{1}{3}\cosh(2\tau + 2r)\sin^2(\zeta\tau)\right] \tag{47}$$

$$\mathcal{E}_3(\tau) \approx \vartheta_1[\cosh(2\tau + 2r)\cos^2(\zeta\tau) + 3\cosh(2\tau - 2r)\sin^2(\zeta\tau)] \tag{48}$$

Now, at the moments close to $\tau_n = \pi n/\zeta$ we observe minima of $\mathcal{E}_1$ (which can be much less than the initial energy, if $r \gg 1$) and maxima of $\mathcal{E}_3$, while the picture is inverted at the moments close to $\tau_n' = (n + 1/2)\pi/\zeta$: see figure 2. The inverse negativity is less sensitive to the phases χ_1 and χ_2. In particular, the intervals of time when $\mathcal{I}(\tau) < 0$ almost do not depend on concrete values of χ_1 and χ_2. Only the 'degree of inseparability' is sensitive to the phases (if ζ is not extremely big), due to the contribution of term $Z\sin(2\zeta\tau)/\zeta$ in equation (41), whose sign at the moments $\tilde{\tau}_k = \pi(k + 1/4)/\zeta$ depends on the parity of integer k. This is clearly seen if one compares the insertions of figures 1 and 2. Again we note that transitions from separable to inseparable states are not correlated with the behaviour of mode energies.

6. Conclusion and discussion

Let us list the main results of the paper. First, we have obtained a simple form of Simon's separability criterion for bipartite closed quantum systems, whose evolution is governed by arbitrary time-dependent quadratic Hamiltonians, showing that the separability dynamics of such systems is completely determined by the time dependence of the determinant of the cross-covariance matrix. Second, we have analysed the time evolution of the new separability coefficient ('inverse negativity') for several examples of Hamiltonians, describing parametric conversion, amplification and mixed process of amplification–conversion, which corresponds to the case of a specific (e.g., cubic) cavity with resonantly oscillating boundary. It is worth noting in this connection that time-dependent problems are isomorphic to the problems of transformation of quantum states by some 'active' and 'passive' multiport optical devices, such as beam splitters, interferometers or multiwave mixers [41]. Recently, these devices were studied from the point of view of generating entangled states [42]. Thus, the analysis of the time-dependent problem can easily be reformulated in terms of multiport optical devices after small changes in terminology and redefinition of the meaning of some parameters. In particular, we have confirmed the known result that two modes can be entangled with the aid of a beam splitter (which is equivalent to some parametric converter), only if at least one of them was initially in a nonclassical (squeezed) state, with big enough squeezing coefficient.

A striking result of our study is that transitions from separable to inseparable states are not accompanied by any visible qualitative change in the behaviour of observable physical quantities, such as, e.g., mean energies of modes, or purities of states of each mode (cf [43], where analogous conclusions were made with respect to the degree of 'nonclassicality' of quantum systems). One could think that mean energies of each mode hardly have a direct relation to the correlations between the modes. But a similar behaviour is observed for the correlation between photon numbers of coupled modes

$$\mathcal{K} = \langle(\hat{N}_1 - \langle\hat{N}_1\rangle)\langle(\hat{N}_2 - \langle\hat{N}_2\rangle)\rangle \equiv \langle\hat{N}_1\hat{N}_2\rangle - \langle\hat{N}_1\rangle\langle\hat{N}_2\rangle \qquad \hat{N}_k \equiv \hat{a}_k^\dagger\hat{a}_k.$$

Actually, $\mathcal{K}$ depends on the moments of the first, second, third and fourth orders of the raising and lowering operators. However, for *Gaussian states* all higher order moments can be expressed in terms of the first- and second-order ones. For example, the fourth-order centralized moments of any two real commuting quadratures z_i and z_j (where each z_k may be either x_k or p_k) can be written as [19]

$$\langle(\delta\hat{z}_i)^2(\delta\hat{z}_j)^2\rangle = \langle(\delta\hat{z}_i)^2\rangle\langle(\delta\hat{z}_j)^2\rangle + 2\langle\delta\hat{z}_i\delta\hat{z}_j\rangle^2 \qquad \delta\hat{z}_k \equiv \hat{z}_k - \langle\hat{z}_k\rangle.$$

Thus, for Gaussian states we have

$$\mathcal{K} = |\langle\delta\hat{a}_1\delta\hat{a}_2\rangle|^2 + \left|\langle\delta\hat{a}_1\delta\hat{a}_2^\dagger\rangle\right|^2 + 2\,\mathrm{Re}\left[\langle\delta\hat{a}_1\delta\hat{a}_2\rangle\langle\hat{a}_1\rangle^*\langle\hat{a}_2\rangle^* + \langle\delta\hat{a}_1\delta\hat{a}_2^\dagger\rangle\langle\hat{a}_1\rangle^*\langle\hat{a}_2\rangle\right]. \tag{49}$$

The photon number correlation depends on variances and mean values of the creation and annihilation operators, while the separability properties depend on variances only. If $\langle\hat{a}_k\rangle = 0$, then $\mathcal{K}$ becomes identical (up to a normalization factor) with the *trace–covariance correlation coefficient* introduced in [23, 44]. The time evolution of this quantity for different initial states and different quantum systems was considered in [23, 26, 37]. Analysing the results of that papers, we can conclude that nothing happens with the coefficient $\mathcal{K}$ when separable states of two modes are transformed into inseparable states. Moreover, $\mathcal{K}$ is always nonnegative for $\langle\hat{a}_k\rangle = 0$, independently of the sign of separability coefficients. Although $\mathcal{K}$ can become negative if $\langle\hat{a}_k\rangle \neq 0$, its negativity seems to have a little in common with inseparability (understood as a 'nonclassicality' of the state). For example, in the case of a parametric converter (section 3), for the initial shifted thermal state of the first mode ($r_1 = 0$) and nonshifted thermal state of the second mode ($r_2 = \langle\hat{a}_2(0)\rangle = 0$) we obtain

$$\mathcal{K} = \tfrac{1}{4}\sin^2(2\tau)(\nu_1 - \nu_2)(\nu_1 - \nu_2 + 2|\langle\hat{a}_1(0)\rangle|^2).$$

If the value of $|\langle\hat{a}_1(0)\rangle|^2$ is big enough and $\nu_2 > \nu_1$ (i.e., the temperature of the nonshifted mode is higher than the temperature of the shifted one), then $\mathcal{K} < 0$ for $\tau > 0$, although the joint state of two modes is separable, according to results of section 3.

In this connection, it would be interesting to find physical observables whose behaviour is different for separable and inseparable states of continuous variable systems. Otherwise, we have to admit that the concept of separability, being useful for quantum systems in finite-dimensional Hilbert spaces (in view of applications to quantum computing and quantum information theory), perhaps, loses its significance for continuous variable quantum systems, where such quantities as energies or other directly measurable observables play more important roles.

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